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Nonholonomic deformation of generalized KdV-type equations

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Abstract
Karasu-Kalkani *et al* (2008 *J. Math. Phys.* **49** 073516) recently derived a new sixth-order wave equation KdV6, which was shown by Kupershmidt (2008 *Phys. Lett.* **372A** 2634) to have an infinite commuting hierarchy with a common infinite set of conserved densities. Incidentally, this equation was written for the first time by Calogero and is included in the book by Calogero and Degasperis (1982 *Lecture Notes in Computer Science* vol 144 (Amsterdam: North-Holland) p 516). In this paper, we give a geometric insight into the KdV6 equation. Using Kirillov's theory of coadjoint representation of the Virasoro algebra, we show how to obtain a large class of KdV6-type equations equivalent to the original equation. Using a semidirect product extension of the Virasoro algebra, $\widehat{\mathrm{Vir} \ltimes C^\infty(S^1)}$, we propose the nonholonomic deformation of the Ito equation. We also show that the Adler–Kostant–Symes scheme provides a geometrical method for constructing nonholonomic deformed integrable systems. Applying the Adler–Kostant–Symes scheme to loop algebra, we construct a new nonholonomic deformation of the coupled KdV equation.

Dedicated to Professor N Mukunda on his 70th birthday

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1. Introduction

Recently, Karasu-Kalkani *et al* [12] applied the Painlevé test to the class of sixth-order nonlinear wave equations and found that three of these were previously known, but the fourth one turned out to be a new one:

$$\left(\partial_x^2 + 8u_x\partial_x + 4u_{xx}\right)\left(u_t + u_{xxx} + 6u_x^2\right) = 0. \tag{1}$$

One immediately recognizes the potential form of the KdV equation in the second factor of the left-hand side of (1). The factored way of writing the KdV6 equation has the advantage of the fact that all solutions of (potential) KdV are also solutions of KdV6. It is important to note that equation (1) was written for the first time by Calogero and is contained in the book by Calogero and Degasperis [5]. As such, it shares many of the properties of the equations associated with the Schrödinger spectral problem discussed there.

After a slight change of variables $v = u_x$, $w = u_t + u_{xxx} + 6u_x^2$, equation (1) boils down to

$$v_t + v_{xxx} + 12vv_x - w_x = 0, \qquad w_{xxx} + 8vw_x + 4wv_x = 0. \tag{2}$$

The authors of [12] obtained the Lax pair and an auto-Bäcklund transformation for (2). They claimed that (2) is different from the KdV equation with self-consistent sources (KdVESCS) and posed an open question to find higher symmetries and asked if higher conserved densities and a Hamiltonian formalism exist for (2). In a recent paper, Ramani *et al* [17] bilinearized the KdV6 equation and derived a new, simpler, auto-Bäcklund transformation; starting from the solutions to the KdV equation, we construct solutions to KdV6 in the form of M kinks and N poles which indeed involve an arbitrary function of time.

In an interesting paper, Kupershmidt [14] described this as a *nonholonomic* deformation of the KdV equation. By rescaling v and t, he further modified this to

$$u_t - 6uu_x - u_{xxx} + w_x = 0, \qquad w_{xxx} + 4uw_x + 2u_x w = 0. \tag{3}$$

This can be converted into a bi-Hamiltonian form:

$$u_t = B^1\left(\frac{\delta H_{n+1}}{\delta u}\right) - B^1(w) = B^2\left(\frac{\delta H_n}{\delta u}\right) - B^1(w), \qquad B^2(w) = 0, \tag{4}$$

where

$$B^1 = \partial = \partial_x, \qquad B^2 = \partial^3 + 2(u\partial + \partial u) \tag{5}$$

are the two standard Hamiltonian operators of the KdV hierarchy, $n = 2$, and

$$H_1 = u, \qquad H_2 = u^2/2, \qquad H_3 = u^3/3 - u_x^2/2, \ldots \tag{6}$$

are the conserved densities.

The soliton equations with self-consistent sources have many physical applications; for example, they describe the interaction of long and short capillary-gravity waves. In a recent paper, Yao and Zeng [21] showed that the KdV6 equation is equivalent to the Rosochatius deformation of the KdV equation with self-consistent sources. In this paper, we extend the Yao and Zeng result to construct many other equations equivalent to the KdV6 equation. We identify the fact that the constraint equation of w is a stabilizer equation of the Virasoro orbit. We tacitly replace this equation with an equivalent partner equation to obtain a new *avatar* of the KdV6 equation. Essentially, Yao and Zeng adopted this philosophy in an *ad hoc* style. We put it in a more systematic form using Kirillov's coadjoint orbit method. Our next task is to extend Kupershmidt's formalism to extended the Virasoro algebra $\widehat{\mathrm{Vir} \ltimes C^\infty(S^1)}$ to construct the Ito6 equation. It is known [8, 9] that a wide class of coupled KdV equations can be manifested as geodesic flows of the right invariant L^2 metric on the semidirect product group $\widehat{\mathrm{Diff}(S^1) \ltimes C^\infty(S^1)}$, where $\mathrm{Diff}(S^1)$ is the group of orientation-preserving diffeomorphisms on a circle. We construct nonholonomic deformation of the Ito system from the coadjoint representation of the extended Virasoro algebra [8, 9].

In the second part of the paper, we give a construction of the KdV6 equation using loop algebra. It is well known that a systematic procedure of obtaining the most finite-dimensional completely integrable systems is obtained from the Adler, Kostant and Symes (AKS) theorem

[1, 2, 20] applying to some Lie algebra $\mathfrak{g}$ equipped with an ad-invariant non-degenerate bilinear form. The AKS method provides us with a Poisson manifold and a hierarchy of commuting Hamiltonians. When one applies this scheme to Lie algebras, one obtains discrete integrable systems, for example the open Toda lattice system. However, the most interesting examples are related to infinite-dimensional Lie algebras or loop algebras as shown by Reyman and Semenov-Tian-Sanskii [18, 19]. In this paper, we apply the Adler–Kostant–Symes scheme to loop algebra to give a nonholonomic deformation of the coupled complex KdV equation using the Fordy–Kulish decomposition [6, 7, 16].

The plan of the paper is as follows. We give a preliminary description of the Virasoro algebra and its association with the KdV6 equation in section 2. In section 3, we give various equivalent representations of the KdV6 equation. Section 4 is devoted to the construction of the nonholonomic deformation of the Ito equation, dubbed the Ito6 equation. Using the current algebra method, we derive the nonholonomic deformation of the coupled complex KdV equation. In section 6, we give a brief outlook of the paper.

2. Virasoro algebra and nonholonomic deformation of the KdV equation

Let us consider the Lie algebra of vector fields $\mathrm{Vect}(S^1)$ on a circle S^1. The dual of this algebra is identified with the space of quadratic differential forms $\mathcal{F}_2$. The pairing between $f(x)\frac{\mathrm{d}}{\mathrm{d}x} \in \mathrm{Vect}(S^1)$ and $u(x)\,\mathrm{d}x^{\otimes 2} \in \mathcal{F}_2$ is defined as

$$\left\langle u(x)\,\mathrm{d}x^2, f(x)\frac{\mathrm{d}}{\mathrm{d}x}\right\rangle = \int_0^{2\pi} u(x) f(x)\,\mathrm{d}x.$$

The Virasoro algebra Vir has a unique non-trivial central extension (see, for example, [13]) by means of $\mathbf{R}$:

$$0 \longrightarrow \mathbf{R} \longrightarrow \mathrm{Vir} \longrightarrow \mathrm{Vect}(S^1)$$

described by the Gelfand–Fuchs cocycle

$$\omega_1(f, g) = \int_{S^1} f' g''\,\mathrm{d}x.$$

The elements of Vir can be identified with the pairs (2π periodic function, real number). The commutator in Vir takes the form

$$\left[\left(f(x)\frac{\mathrm{d}}{\mathrm{d}x}, a\right), \left(g(x)\frac{\mathrm{d}}{\mathrm{d}x}, b\right)\right] = \left((fg' - gf')\frac{\mathrm{d}}{\mathrm{d}x}, \int_{S^1} f'g''\,\mathrm{d}x\right).$$

The dual space Vir^* can be identified with the set $\{(\mu, u\,\mathrm{d}x^2) | \mu \in \mathbf{R}\}$.

A pairing between a point $\left(\lambda, f(x)\frac{\mathrm{d}}{\mathrm{d}x}\right) \in \mathrm{Vir}$ and a point $(\mu, u\,\mathrm{d}x^2) \in \mathrm{Vir}^*$ is given by

$$\left\langle (\mu, u(x)\,\mathrm{d}x^2), \left(\lambda, f(x)\frac{\mathrm{d}}{\mathrm{d}x}\right)\right\rangle = \lambda\mu + \int_{S^1} f(x)u(x)\,\mathrm{d}x.$$

Lemma 2.1. *The coadjoint action of the Virasoro algebra* $\left(\lambda, f(x)\frac{\mathrm{d}}{\mathrm{d}x}\right) \in \mathrm{Vir}$ *on its dual* $(\mu, u\,\mathrm{d}x^2) \in \mathrm{Vir}^*$ *is given by*

$$ad^*_{(\lambda, f(x)\frac{\mathrm{d}}{\mathrm{d}x})}(\mu, u\,\mathrm{d}x^2) = \mu f''' + 2f'u + fu'. \tag{7}$$

Proof. It follows from the definition

$$\left\langle ad^*_{(\lambda,f)}(\mu, u), (\nu, g)\right\rangle = \langle(\mu, u), ad_{(\lambda,f)}(\nu, g)\rangle = \left\langle(\mu, u), \left(\int_{S^1} f'g''\,\mathrm{d}x, \left[f\frac{\mathrm{d}}{\mathrm{d}x}, g\frac{\mathrm{d}}{\mathrm{d}x}\right]\right)\right\rangle.$$

$$= \int_{S^1} u(fg' - f'g)\,\mathrm{d}x + \mu\int_{S^1} f'g''.$$

From this expression, we obtain (7). □

We fix the hyperplane $\mu = \frac{1}{2}$. The kernel of ad^* yields the stabilizer set of the Virasoro orbit. It plays a very important role in the definition of the KdV6 equation. Incidentally, the dual space of the Virasoro algebra can also be identified with the space of Hill's operators.

Corollary 2.2. *The stabilizer space of the coadjoint action of* $\left(\lambda, f\frac{\mathrm{d}}{\mathrm{d}x}\right) \in \mathrm{Vir}$ *on the hyperplane* $\left(\mu = \frac{1}{2}\right)$ *of the space of Hill's operator (or quadratic differentials) is given by*

$$f''' + 4u'f + 4uf' = 0 \tag{8}$$

or

$$ff'' + 2uf^2 - \frac{1}{2}(f')^2 = c, \tag{9}$$

where c is a constant.

The second Hamiltonian operator of the KdV equation can be easily derived from the coadjoint action (7), given by

$$\mathcal{O}^2_{\mathrm{KdV}} = D^3 + 4uD + 2u_x, \qquad \text{where} \qquad D = \frac{\mathrm{d}}{\mathrm{d}x}. \tag{10}$$

It is known that the first Hamiltonian operator of the KdV equation can also be derived (see below for the derivation of this operator for the Ito system) from the coadjoint action using the frozen Lie–Poisson structure. It is given by $\mathcal{O}^1_{\mathrm{KdV}} = D$.

Proposition 2.3. *The KdV6 equation is the constraint Hamiltonian flow on the Virasoro orbit:*

$$u_t = ad^*_{\nabla H}(u) - w_x = \mathcal{O}^2_{\mathrm{KdV}}\frac{\delta H}{\delta u} - \mathcal{O}^1_{\mathrm{KdV}}(w) \qquad s.\ t. \qquad \langle \nabla H, w_x\rangle = 0 \tag{11}$$

and

$$\mathcal{O}^2_{\mathrm{KdV}}(w) = 0, \tag{12}$$

where $H = \frac{1}{2}\int_{S^1} u^2\,\mathrm{d}x$.

This definition is closely related to the Euler–Poincaré–Suslov (EPS) formalism [3, 11]. In fact, this is one of the best demonstrations of the EPS formalism in integrable systems. Given a set of linearly independent vectors a^i, this nonholonomic flow equation can be expressed in canonical coordinates $x = (p, q)$:

$$\dot{x} = [x, \nabla H(x)] + \sum_{i=1}^{p} \lambda_i a^i,$$

such that

$$\langle \nabla H(x), a^i\rangle = 0, \qquad i = 1, \ldots, p.$$

Our next task is to find w such that it satisfies $\langle \nabla H, w_x\rangle = 0$. The workable approach to this problem is to choose Kupershmidt's scheme, i.e.

$$w = \frac{\delta G}{\delta u}, \tag{13}$$

for some function G. This immediately leads to

$$\langle \nabla H, w_x\rangle = \left\langle \frac{\delta H}{\delta u}, \partial\left(\frac{\delta G}{\delta u}\right)\right\rangle = \{H, G\}_1 = 0.$$

Since $\mathcal{O}_2 w = 0$, hence

$$\{H, G\}_2 = 0.$$

Thus, G commutes with H with respect to both the Poisson structures. It is easy to generalize this construction to the sequence of Hamiltonians H_n. Thus, G commutes with H_n w.r.t. to both the brackets, i.e.

$$\{H_n, G\}_1 = 0 = \{H_n, G\}_2. \tag{14}$$

3. Various equivalent forms of the KdV6 equation

In this section, we study various equivalent forms of the nonholonomic deformation of the KdV equation. It is known that the KdV6 equation always appears as a pair of equations (3), an evolution equation of u and a constraint equation of w. In principle, the second equation of (3), i.e. w equation, can always be written in various equivalent forms and the first equation must be replaced by new 'w'. We illustrate this formalism by examples below. Let us start with an example discussed by Yao and Zeng [21], where the w equation is replaced by Ermakov–Pinney-type systems.

3.1. Ermakov–Pinney-type systems

The classical Pinney equation may be regarded as the simplest specialization of a pair of coupled second-order ordinary differential equations, now known as Ermakov systems. The Ermakov systems usually belong to a general class of equations of the form

$$p_i'' + up_i = g_i(p_1, \ldots, p_n), \qquad i = 1, \ldots, n,$$

where g_i are homogeneous functions of weight -3 and $u = u(x, p_1, \ldots, p_n, p_1', \ldots, p_n')$.

The simplest Ermakov system reads as

$$\psi'' + u(x)\psi = \frac{\sigma}{\psi^3}. \tag{15}$$

We need some properties of the w (or stabilizer) equation to understand this change. We start with an important property of the w equation. Its solution can be manufactured from Hill's equation.

It is straightforward to see that if ψ_1 and ψ_2 are the solutions of Hill's equation

$$\Delta\psi = \left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + u\right)\psi = 0, \tag{16}$$

then the product $w = \psi_i\psi_j$ satisfying the constraint (stabilizer) equation $w'''+2u'w+4uw' = 0$ traces out a three-dimensional space of solution.

Since the solution of the constraint equation is spanned by $\psi_i\psi_j$, i.e.

$$\mathrm{Span}\big(\psi_1^2, \psi_2^2, \psi_1\psi_2\big),$$

naturally, an arbitrary solution of w is given by

$$\Psi = A\psi_1^2 + 2B\psi_1\psi_2 + C\psi_2^2, \tag{17}$$

an arbitrary linear combination of basis vectors. This is periodic and hence a global solution of the constraint equation. We skip the geometric meaning of Ψ in this paper; the interested reader can refer to [10]. The square root of Ψ is connected to the solution of the Ermakov equation. We state this result below.

Proposition 2. *If ψ_1 and ψ_2 satisfy Hill's equation, then*

$$\psi = \sqrt{A\psi_1^2 + 2B\psi_1\psi_2 + C\psi_2^2} \tag{18}$$

satisfies the Ermakov equation

$$\psi'' + u(x)\psi = \frac{\sigma}{\psi^3}, \qquad \sigma = AC - B^2$$

and $\left\{\psi_1^2, \psi_2^2, \psi_1\psi_2\right\}$ satisfy the stabilizer or constraint equation $w''' + 4uw' + 2u'w = 0$.

This proof follows from direct calculation. This proposition allows us to state that if we set $w = \psi^2$, then the constraint equation is the Ermakov equation. Essentially, this is the important observation of Yao and Zeng [21].

We can associate another interesting equation with the w equation, which is closely related to the Ermakov–Pinney equation. This is known as the Kummer–Schwarz equation:

$$\frac{1}{2}\frac{w''}{w} - \frac{3}{4}\left(\frac{w'}{w}\right)^2 + \sigma w^2 = u(x).$$

The solution of this equation is given by

$$w(x) = \left(A\psi_1^2 + 2B\psi_1\psi_2 + C\psi_2^2\right)^{-1}, \tag{19}$$

where ψ_1 and ψ_2 satisfy Hill's equation.

Thus by establishing the connection between the solutions of the w equation and the Ermakov–Pinney equation, we replace the old set of KdV6 pair with a new set of KdV6 pair. This is an equivalent representation of the KdV6 equation. We give another example below in the spirit of Yao and Zeng [21].

3.2. *Second-order Riccati equation*

The objective of this section is to present another version of the KdV6 equation. Here, the w equation is replaced by the second-order Riccati equation.

Let L be the following differential operator:

$$L = \frac{\mathrm{d}}{\mathrm{d}x} + v(x).$$

The nth-order equation of the Riccati chain is given by the following formula:

$$L^n v(x) + \sum_{j=1}^{n-1} \alpha_j(x)(L^{j-1}v(x)) + \alpha_0(x) = 0, \tag{20}$$

where n is an integer characterizing the order of the Riccati equation in the chain and $\alpha_j(x)$, $j = 0, 1, \ldots, n$, are arbitrary functions. In particular, the second-order Riccati equation (SORE) is given by

$$n = 2, \qquad v_{xx} + 3v(x)v_x + v^3(x) + \alpha_1(x)v(x) + \alpha_0(x) = 0. \tag{21}$$

Now we wish to establish a connection between the w (or stabilizer) equation and SORE. Suppose that

$$v = \frac{w_x}{w}; \tag{22}$$

then we obtain

$$\frac{w_{xxx}}{w} = v_{xx} + 3vv_x + v^3.$$

After substituting this into $w_{xxx} + 4uw_x + 2u_x w = 0$, we find

$$v_{xx} + 3vv_x + v^3 + 4uv + 2u_x = 0, \tag{23}$$

which is a particular case of the second-order Riccati equation. It is now clear that the w equation is replaced by (23).

There are some interesting properties of (23). In fact, the solution of the second-order Riccati equation can be obtained from the ordinary Riccati equation. Let us seek to find the relation between the solutions of the ordinary Riccati equation and the second-order Riccati equation (23).

Proposition 3.2.

(i) The projective vector field equation is equivalent to a particular form of the second-order Riccati equation $v_{xx} + 3vv_x + v^3 + 4uv + 2u_x = 0$*, where* $v = f_x/f$*.*

(ii) Suppose that $v(x) = v_1$ *is the solution of the Riccati equation* $v_x + v^2 + u = 0$*. Then the second-order Riccati equation satisfies* $v(x) = 2v_1$*.*

Proof. By direct computation, one can check this result. □

Thus, we see that the w equation can be replaced by the second-order Riccati equation. In this way, we can obtain several equivalent representations of the KdV6 equation. We list them systematically.

Equivalent representation of the KdV6 equation

The KdV6 equation is equivalent to the following sets of equations.

(i) Let ψ_1 and ψ_2 be the solutions of Hill's equation. Then the solution of the constraint equation is $w = \psi_i\psi_j$. When we replace the constraint equation of w by Hill's equation, then KdV6 is given by

$$u_t = u_{xxx} + 6uu_x - (\psi_i\psi_j)_x, \qquad \psi_{xx} + u\psi = 0. \tag{24}$$

(ii) If the constraint equation is replaced by the second-order Riccati equation, then the KdV6 equation becomes

$$u_t = u_{xxx} + 6uu_x - (\mathrm{e}^{\int^x v\,\mathrm{d}x'})_x, \qquad v_{xx} + +3vv_x + v^3 + 4uv_x + 2u_xv = 0, \tag{25}$$

where $v = w_x/w$.

(iii) If the constraint equation is the Ermakov equation, then the new KdV6 equation is

$$u_t = u_{xxx} + 6uu_x - (\psi^2)_x, \qquad \psi_{xx} + u\psi = \frac{\sigma}{\psi^3}, \tag{26}$$

where $w = \psi^2$ and ψ satisfy equation (18).

4. Generalizing KdV6 to the Ito6 equation

We now generalize the KdV6 equation to the multi-component KdV6 equation. We use a semidirect product extension of the Virasoro algebra [8, 9] to construct the Ito6-type system. Let us give a quick introduction to the semidirect extension of the Bott–Virasoro group and the corresponding Virasoro algebra.

The Lie algebra of $\mathrm{Diff}(S^1) \ltimes C^\infty(S^1)$ is the semidirect product Lie algebra:

$$\mathcal{G} = \mathrm{Vect}(S^1) \ltimes C^\infty(S^1).$$

An element of $\mathcal{G}$ is a pair $\left(f(x)\frac{\mathrm{d}}{\mathrm{d}x}, a(x)\right)$, where $f(x)\frac{\mathrm{d}}{\mathrm{d}x} \in \mathrm{Vect}(S^1)$ and $a(x) \in C^\infty(S^1)$.

It is known that this algebra has a three-dimensional central extension given by the non-trivial cocycles

$$\omega_1\left(\left(f\frac{\mathrm{d}}{\mathrm{d}x}, a\right), \left(g\frac{\mathrm{d}}{\mathrm{d}x}, b\right)\right) = \int_{S^1} f'(x)g''(x)\,\mathrm{d}x \tag{27}$$

$$\omega_2\left(\left(f\frac{\mathrm{d}}{\mathrm{d}x}, a\right), \left(g\frac{\mathrm{d}}{\mathrm{d}x}, b\right)\right) = \int_{S^1} f''(x)b(x) - g''a(x))\,\mathrm{d}x \tag{28}$$

$$\omega_3\left(\left(f\frac{\mathrm{d}}{\mathrm{d}x}, a\right), \left(g\frac{\mathrm{d}}{\mathrm{d}x}, b\right)\right) = 2\int_{S^1} a(x)b'(x)\,\mathrm{d}x. \tag{29}$$

We consider an extension of $\mathcal{G}$. This extended algebra is given by

$$\hat{\mathcal{G}} = \mathrm{Vect}(S^1) \ltimes C^\infty(S^1) \oplus \mathbb{R}^3. \tag{30}$$

Definition 4.1. *The commutation relation in* $\hat{\mathcal{G}}$ *is given by*

$$\left[\left(f\frac{\mathrm{d}}{\mathrm{d}x}, a, \alpha\right), \left(g\frac{\mathrm{d}}{\mathrm{d}x}, b, \beta\right)\right] := \left((fg' - f'g)\frac{\mathrm{d}}{\mathrm{d}x}, fb' - ga', \omega\right), \tag{31}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\beta = (\beta_1, \beta_2, \beta_3), \omega \in \mathbb{R}^3 = (\omega_1, \omega_2, \omega_3)$ *are the 2-cocycles.*

The dual space of smooth functions $C^\infty(S^1)$ is the space of distributions (generalized functions) on S^1. Of particular interest are the orbits in $\hat{\mathcal{G}}^*_{\mathrm{reg}}$. In the case of the current group, Gelfand, Vershik and Graev have constructed some of the corresponding representations.

Definition 4.2. *The regular part of the dual space* $\hat{\mathcal{G}}^*$ *to the Lie algebra* $\hat{\mathcal{G}}$ *is as follows. Consider*

$$\hat{\mathcal{G}}^*_{\mathrm{reg}} = C^\infty(S^1) \oplus C^\infty(S^1) \oplus \mathbb{R}^3$$

and fix the pairing between this space and $\hat{\mathcal{G}}$, $\langle\cdot,\cdot\rangle : \hat{\mathcal{G}}^*_{\mathrm{reg}} \otimes \hat{\mathcal{G}} \to \mathbb{R}$*:*

$$\langle \hat{u}, \hat{f}\rangle = \int_{S^1} f(x)u(x)\,\mathrm{d}x + \int_{S^1} a(x)v(x)\,\mathrm{d}x + \alpha\cdot\gamma, \tag{32}$$

where $\hat{u} = (u(x), v, \gamma)$, $\hat{f} = \left(f\frac{\mathrm{d}}{\mathrm{d}x}, a, \alpha\right)$.

The three following elements

$$\hat{f} = \left(f\frac{\mathrm{d}}{\mathrm{d}x}, a, \alpha\right), \qquad \hat{g} = \left(g\frac{\mathrm{d}}{\mathrm{d}x}, b, \beta\right), \qquad \hat{u} = \left(u\frac{\mathrm{d}}{\mathrm{d}x}, v, c\right)$$

are given in $\hat{\mathcal{G}}$.

Lemma 4.3.

$$ad^*_{\hat{f}}\hat{u} = \begin{pmatrix} (2f'(x)u(x) + f(x)u'(x) + a'v(x) - c_1 f''' + c_2 a'' \\ f'v(x) + f(x)v'(x) - c_2 f''(x) + 2c_3 a'(x) \\ 0 \end{pmatrix}.$$

Proof. This follows from

$$\begin{aligned}
\left\langle ad^*_{\hat{f}}\hat{u}, \hat{g}\right\rangle_{L^2} &= \langle \hat{u}, [\hat{f}, \hat{g}]\rangle_{L^2} \\
&= \left\langle \left(u(x)\frac{\mathrm{d}}{\mathrm{d}x}, v(x), c\right), \left[(fg' - f'g)\frac{\mathrm{d}}{\mathrm{d}x}, fb' - ga', \omega\right]\right\rangle_{L^2} \\
&= -\int_{S^1}(fg' - f'g)u(x)\,\mathrm{d}x - \int_{S^1}(fb' - ga')v\,\mathrm{d}x - c_1\int_{S^1} f'(x)g''(x)\,\mathrm{d}x \\
&\quad - c_2\int_{S^1}(f''(x)b(x) - g''(x)a(x))\,\mathrm{d}x - 2c_3\int_{S^1} a(x)b'(x)\,\mathrm{d}x.
\end{aligned}$$

Since f, g, u are periodic functions, hence integrating by parts we obtain

$$\begin{aligned}
\mathrm{rhs} = \langle 2f'(x)u(x) &+ f(x)u'(x) + a'(x)v(x) - c_1 f'''(x) \\
&+ c_2 a''(x), f'(x)v(x) + f(x)v'(x) - c_2 f'' b(x) + 2c_3 a'(x), 0\rangle.
\end{aligned}$$

□

Consider the following 'modified 'Gelfand–Fuchs' cocycle on $\mathrm{Vect}(S^1)$:

$$\omega_{\mathrm{mGF}}\left(f(x)\frac{\mathrm{d}}{\mathrm{d}x}, g(x)\frac{\mathrm{d}}{\mathrm{d}x}\right) = \int_{S^1}(af'g'' + bf'g)\,\mathrm{d}x. \tag{33}$$

This cocycle is cohomologous to the Gelfand–Fuchs cocycle; hence, the corresponding central extension is isomorphic to the Virasoro algebra. The additional term in (33) is a coboundary term. It is easy to check that the functional

$$\int_{S^1} f' g \, \mathrm{d}x = \frac{1}{2} \int_{S^1} (f' g - f g') \, \mathrm{d}x$$

depends on the commutator of $f \frac{\mathrm{d}}{\mathrm{d}x}$ and $g \frac{\mathrm{d}}{\mathrm{d}x}$.

Taking the modified Gelfand–Fuchs cocycle into account, the Hamiltonian structure associated with the (modified) coadjoint action yields

$$\mathcal{O} = \begin{pmatrix} -c_1 D^3 + 2uD + u_x + c_4 D & vD + c_2 D^2 \\ v_x + vD - c_2 D^2 & 2c_3 D \end{pmatrix}. \tag{34}$$

4.1. Ito6 equation

In this section, we derive one of the main results of the paper, i.e. the nonholonomic deformation of the Ito system, dubbed as the Ito6 equation. This can be considered as a multi-component generalization of the KdV6 equation.

The Hamiltonian structures of the well-known Ito system

$$\begin{aligned} u_t &= u_{xxx} + 6uu_x + 2vv_x \\ v_t &= 2(uv)_x \end{aligned}$$

are given by

$$\mathcal{O}^2_{\text{Ito}} = \begin{pmatrix} D^3 + 4uD + 2u_x & 2vD \\ 2v_x + 2vD & 0 \end{pmatrix} \tag{35}$$

and

$$\mathcal{O}^1_{\text{Ito}} = \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix}. \tag{36}$$

These two Hamiltonian structures can be easily derived from equation (34). Let us choose the *hyperplane* in the dual space $\left(\widehat{\mathrm{Vect}(S^1) \ltimes C^\infty}(S^1)\right)^*$. The coadjoint action leaves the parameter space invariant. When we consider a hyperplane $c_1 = -1, c_2 = c_3 = c_4 = 0$, we obtain the second Hamiltonian structure and for $c_1 = c_2 = 0, c_3 = \frac{1}{2}, c_4 = 1$ we obtain the first Hamiltonian structure.

Proposition 4.4. *The Ito6 equation is a constraint flow on the dual space of the semidirect algebra $\hat{\mathcal{G}}$ restricted to hyperplane $c_1 = -1, c_2 = c_3 = c_4 = 0$:*

$$u_t = u_{xxx} + 6uu_x + 2vv_x - w_{1x} \tag{37}$$

$$v_t = 2(uv)_x - w_{2x}, \tag{38}$$

where $\mathbf{w} = \binom{w_1}{w_2}$ satisfies

$$w_{1xxx} + 6uw_{1x} + vw_{2x} + w_{2xx} = 0 \tag{39}$$

$$-w_{2xx} + (w_1 v)_x = 0. \tag{40}$$

Proof. We use the Euler–Poincaré–Suslov-type equation

$$\begin{pmatrix} u \\ v \end{pmatrix} = \mathcal{O}^2_{\mathrm{Ito}} \begin{pmatrix} \frac{\delta H}{\delta u} \\ \frac{\delta H}{\delta v} \end{pmatrix} - \mathcal{O}^2_{\mathrm{Ito}} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

and constraint equation

$$\mathcal{O}^2_{\mathrm{Ito}} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 0$$

to produce our result. □

Following Kupershmidt, we can show the existence of an infinite number of conserved densities:

$$\frac{\mathrm{d}H_m}{\mathrm{d}t} = \nabla H_m(\mathbf{u})\big[\mathcal{O}^2_{\mathrm{Ito}}\big(\nabla H_n(\mathbf{u}) - \mathcal{O}^1_{\mathrm{Ito}}(\mathbf{w})\big)\big] = \nabla H_m(\mathbf{u})\mathcal{O}^2_{\mathrm{Ito}}(\nabla H_n \mathbf{u}) - \nabla H_n\big(\mathcal{O}^1_{\mathrm{Ito}}\big)\mathbf{w} = 0,$$

where all the operations are defined up to exact differential and

$$\nabla H_m(\mathbf{u}) = \begin{pmatrix} \frac{\delta H_m}{\delta u} \\ \frac{\delta H_m}{\delta v} \end{pmatrix}.$$

5. Loop algebra and nonholonomic deformation of the coupled KdV equation

The AKS theory produces hierarchies of completely integrable partial (or ordinary) differential equations. This scheme is in quite a general framework and based on the following ingredients.

(a) A Lie algebra $\mathfrak{g}$, with a non-degenerate bilinear form $\langle ., \rangle$, allows us to identify $\mathfrak{g}$ with its dual $\mathfrak{g}^*$. The Lie algebra $\mathfrak{g}$ splits $\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}^-$ into two subalgebras $\mathfrak{g}^+$ and $\mathfrak{g}^-$. The bilinear form is used to identify $\mathfrak{g}^{-*}$ with $\mathfrak{g}^{+\perp}$.

(b) The phase space is an ad^*-invariant finite-dimensional submanifold $\Gamma \subset \mathfrak{g}^{-*} \equiv \mathfrak{g}^{+\perp}$. The Poisson structure on Γ is the Kostant–Kirillov structure associated with $\mathfrak{g}^{-*}$.

(c) The complete set of commutating constants of motion will be elements of the algebra $A(\Gamma)$ of ad-invariant functions on $\mathfrak{g}^*$ restricted to Γ.

5.1. Application to the loop group

Let us apply this scheme to the loop group. Let ΩG be the space of the based loop; then the corresponding Lie algebra is called the loop algebra of the semi-infinite formal Laurent series in the variable λ with coefficients in $\mathfrak{g}$:

$$\Omega\mathfrak{g} = \left\{ X(\lambda) = \sum_i x_i \lambda^i ; x_i \in \mathfrak{g} \right\},$$

with the Lie bracket

$$[X(\lambda), Y(\lambda)] := \sum_{i,j} [x_i, y_j]\lambda^{i+j}, \qquad \text{where} \qquad X(\lambda) = \sum x_i \lambda^i, \qquad Y(\lambda) = \sum y_j \lambda^j.$$

Here, we can define the projection operator in the following way:

$$P_\pm X = \begin{cases} X & \text{if } \; X = \sum_{n\geqslant 0} X_n \lambda^n \\ -X & \text{if } \; X = \sum_{n<0} X_n \lambda^n \end{cases}.$$

We define the bilinear form on $\Omega\mathfrak{g}$ as

$$\langle X(\lambda), Y(\lambda)\rangle := \mathrm{Res}_{\lambda=0}\, \mathrm{tr}(\lambda^{-1}(X(\lambda)Y(\lambda)) = \mathrm{tr}(X(\lambda)Y(\lambda))_0.$$

There is a natural splitting in the loop algebra, and the two subalgebras of $\Omega\mathfrak{g}$ are given as

$$\Omega\mathfrak{g}_+ := \left\{ \sum_0^k g_i \lambda^i : g_i \in \mathfrak{g} \right\}, \qquad \Omega\mathfrak{g}_- := \left\{ \sum_{-\infty}^{-1} g_i \lambda^i : g_i \in \mathfrak{g} \right\}.$$

The above decomposition of $\Omega\mathfrak{g}$ does not correspond to the global decomposition of the loop group ΩG, but we have a dense open subset

$$\Omega G^- \Omega G^+ \subset \Omega G \tag{41}$$

consisting of all loops ϕ that can be factorized in the form

$$\phi = \phi^- \phi^+ \tag{42}$$

with $\phi^- \in \Omega^- G$, $\phi^+ \in \Omega^+ G$. We refer to this subset of ΩG as the *big cell*.

With the above choice of the inner product, one can easily verify $\Omega\mathfrak{g}_-{}^* = \Omega\mathfrak{g}_+{}^\perp$, so that Γ can be identified with a submanifold of $\Omega\mathfrak{g}_+{}^\perp$:

$$\Gamma := \{A(\lambda) = \sum_0^n a_{n-i} \lambda^i, n \text{ fixed }\}.$$

The Kostant–Kirillov bracket for $\widehat{\Omega\mathfrak{g}}^*$ is given by

$$\{f, g\}(\mu) = \langle \mu, [\nabla f(\mu), \nabla g(\mu)] \rangle, \qquad \text{where} \qquad \mu \in \Omega\mathfrak{g}^*. \tag{43}$$

The gradient of a function $f : \mathfrak{g}^* \longrightarrow \mathbf{C}$ is the vector field $\nabla f : \mathfrak{g}^* \longrightarrow \mathfrak{g}$ such that

$$\langle \nabla f(\mu), X(\mu) \rangle = \mathrm{d} f(X(\mu)) \qquad \forall \mu \in \mathfrak{g}^*.$$

But this does not restrict to $\Omega\mathfrak{g}^*_-$. In fact, with respect to this bracket, the Hamiltonian vector fields of elements of $A(\Gamma)$ are identically zero; one justifies this by $\langle ad^*_X \mu, \nabla H \rangle = \langle \mu, [X, \nabla H] \rangle = 0$ for all $X \in \Omega\mathfrak{g}$.

Let us consider the Hamiltonian equation with respect to $\{.,.\}$ where H can be expressed in terms of linear coordinates $\mu_r = \langle \mu, X_r \rangle$, where X_r form the basis in $\Omega\mathfrak{g}_-$. Thus, the Hamiltonian equation becomes

$$\begin{aligned}
\langle \dot{\mu}, X_r \rangle &= \{H, \mu_r\}(\mu) = \langle \mu, [(\nabla H(\mu))_-, (\nabla \mu_r(\mu))_-] \rangle \\
&\Longrightarrow \langle \dot{\mu}, X_r \rangle = \langle \mu, [(\nabla H(\mu))_-, X_r] \rangle \\
&\Longrightarrow \langle \dot{\mu}, X_r \rangle = \langle [(\nabla H(\mu))_+, \mu], X_r \rangle;
\end{aligned}$$

hence, we obtain

$$\dot{\mu} = [(\nabla H(\mu))_+, \mu]. \tag{44}$$

Thus, we derive the AKS equation without the cocycle term.

Our aim is to extend the loop algebra $\Omega\mathfrak{g}$. Let us consider the Grassmannian-like homogeneous space $\Omega G/\Omega G^+$. The image in $\Omega G/\Omega G^+$ of the complement of the big cell in ΩG is a divisor in $\Omega G/\Omega G^+$; it therefore corresponds to a holomorphic line bundle $\mathcal{L}$ over $\Omega G/\Omega G^+$. We denote by $\tilde{\Omega} G$ the automorphism group of $\mathcal{L}$. The pullback of $\mathcal{L}$ to $\hat{\Omega} G$ is canonically trivial. Hence, $\tilde{\Omega} G$ turns out to be the central extension of ΩG by $\mathbf{C}^\times$:

$$1 \longrightarrow \mathbf{C}^\times \longrightarrow \tilde{\Omega G} \longrightarrow \Omega G \longrightarrow 1. \tag{45}$$

Hence, we obtain the central extension corresponding to the Lie algebra:

$$0 \longrightarrow \mathbf{C} \longrightarrow \tilde{\Omega \mathcal{G}} \longrightarrow \Omega \mathcal{G} \longrightarrow 0.$$

We introduce a non-trivial 2-cocycle on $\Omega\mathfrak{g}$, known as the Maurer–Cartan cocycle. Then corresponding to the centrally extended loop group $\widehat{\Omega G} = \Omega G \times \mathbf{C}$, the Lie algebra is $\widehat{\Omega\mathfrak{g}} = \Omega\mathfrak{g} \oplus \mathbf{C}$. This is a centrally extended loop algebra associated with a 2-cocycle:

$$\omega(X, Y) = \left(X, \frac{\mathrm{d}Y}{\mathrm{d}x}\right) = \int_{S^1} X'Y \,\mathrm{d}x. \tag{46}$$

Loop algebra $\widehat{\Omega\mathfrak{g}}$ satisfies the following commutation relation:

$$[(X, a), (Y, b)] = \left([X, Y], \int_{S^1} \mathrm{tr}(XY')\right),$$

where $(X, a), (Y, b) \in \widehat{\Omega\mathfrak{g}}$.

In general, the map

$$\kappa : \Omega\mathfrak{g} \longrightarrow \tilde{\Omega\mathfrak{g}} \equiv \Omega\mathfrak{g} \oplus \mathbf{C} \tag{47}$$

is not a Lie algebra homomorphism; only its restriction to $\Omega\mathfrak{g}^+$ is a Lie algebra homomorphism, since the central extension term vanishes identically. Then the corresponding induced map

$$\kappa : \Omega G^+ \longrightarrow \tilde{\Omega G} \tag{48}$$

yields a canonical holomorphic trivialization of the part of the fibration lying over ΩG^+.

We also define the bilinear form on $\widehat{\Omega\mathfrak{g}}$ by

$$\langle (X, a), (Y, b)\rangle = ab + \int \mathrm{tr}(XY).$$

Suppose that H is an ad-invariant function on $\Omega\mathfrak{g}^*$; then

$$ad^*(\nabla H(\alpha), a)(\mu, 1) = ((ad^*(\nabla H(\mu))(\mu) + (\nabla H)', 0).$$

Theorem 5.1. *Let* $\tilde{\Omega\mathfrak{g}} = \tilde{\Omega\mathfrak{g}}^+ \oplus \tilde{\Omega\mathfrak{g}}^-$ *and* $M \subset \tilde{\Omega\mathfrak{g}}^+$ *be a coadjoint orbit equipped with a natural weak orbit symplectic structure* ω. *Let* $H_i : \tilde{\Omega\mathfrak{g}} \longrightarrow \mathbf{C}$ *be the set of ad-invariant functions in* $I(g^*)$ *restricted to* $(\tilde{\Omega\mathfrak{g}}^+)^\perp$ *is an involutive system on the coadjoint orbit. The Hamiltonian equations of motion on* $\tilde{\Omega\mathfrak{g}}^*$ *generated by the Hamiltonian (ad-invariant function) have the form*

$$\frac{\partial\mu}{\partial t} = \frac{\partial L}{\partial x} + [L, \mu], \tag{49}$$

where $L = \pi_+[\mathrm{grad}\, H]$.

Our aim is to find integrable systems related to Hermitian symmetric spaces as an application of the Adler–Kostant–Symes scheme. Our main interest is in $\mathbf{CP^1} = SU(2)/U(1)$. In general, any semisimple Lie algebra can be decomposed to $\mathfrak{g} = t + m$ such that t is the maximally commutating subalgebra and m is the complement of t in $\mathfrak{g}$. We can identify m with the tangent space of the homogeneous manifold $M = G/T$, where G is the Lie group associated with $\mathfrak{g}$ and T is the subgroup associated with t. When the decomposition satisfies

$$[t, t] \subset t \qquad \text{and} \qquad [t, m] \subset m,$$

$\mathfrak{g}$ is called reductive decomposition.

If in addition to these t, m satisfies one extra condition $[m, m] \subset t$, then it is called the symmetric decomposition of $\mathfrak{g}$ and the space $M = G/T$ is called the Hermitian symmetric space. The Killing form of $\mathfrak{g}$ descends down to give metric on this space. In the case of the Hermitian symmetric space, there exists an element $A \in t$ such that $t = C_{\mathfrak{g}}(A) = \{s \in \mathfrak{g}; [A, s] = 0\}$. Let h be the Cartan subalgebra of $\mathfrak{g}$; the element A can be chosen to lie in h. We have $m = m^+ \oplus m^-$ given by $[A, t] = 0$ and $[A, X^{\pm}] = \pm X^{\pm}$ for all value $X^{\pm} \in m^{\pm}$.

In the following section, we will derive a coupled complex KdV equation and its nonholonomic deformation counterpart from the AKS scheme applied to homogeneous spaces. This is also known as the Fordy–Kulish decomposition [6].

5.2. Complex coupled KdV and KdV equations

First, we will study the construction of KdV and coupled complex KdV equations. This method was applied in [16, 7].

We choose $\mu = \lambda A + Q$. Let us define $\nabla H = \sum_{j=1}^{4} h^j(x, t)\lambda^j$ with $h^j = h^j_m + h^j_k$.

Thus from the AKS equation

$$(\lambda A + Q)_t = [A\lambda + Q, \nabla H] - (\nabla H)_x,$$

we obtain various coefficients of ∇H by setting all λ^m equal to zero.

Thus, we obtain the following relations recursively:

$$h_3 = A, \qquad h_2 = Q, \qquad h_1 = \frac{\mathrm{i}}{2}Q^+_x - \frac{\mathrm{i}}{2}Q^-_x - \frac{\mathrm{i}}{2}[Q^-, Q^+],$$

and

$$h_0 = T + [S, Q],$$

where

$$T = -\tfrac{1}{4}Q_{xx} + \tfrac{1}{4}[Q^+, [Q^-, Q^+]] - \tfrac{1}{4}[Q^-, [Q^-, Q^+]]$$

and

$$S = \tfrac{1}{4}\left(Q^+_x + Q^-_x\right) + c(Q_+ + Q_-).$$

Therefore, the gradient of Hamiltonian ∇H is given by

$$\begin{aligned}\nabla H = A\lambda^3 + Q\lambda^2 &+ \left(\frac{\mathrm{i}}{2}Q^+_x - \frac{\mathrm{i}}{2}Q^-_x - \frac{\mathrm{i}}{2}[Q^-, Q^+]\right)\lambda \\ &+ \left(-\frac{1}{4}Q_{xx} + \frac{1}{4}[Q^+, [Q^-, Q^+]] - \frac{1}{4}[Q^-, [Q^-, Q^+]]\right) \\ &+ \left[\frac{1}{4}(Q^+_x + Q^-_x) + c(Q_+ + Q_-), Q\right].\end{aligned}$$

Finally, equating λ^0 we obtain

$$T_x + [S, Q]_x = [Q, T] + [Q, [S, Q]] + Q_t. \tag{50}$$

Note that this is the zero curvature equation. Let us fix

$$A = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}$$

and if we choose

$$Q = \begin{pmatrix} 0 & q^\dagger \\ -q & 0 \end{pmatrix}, \tag{51}$$

then we can express these two linear pairs of equations as

$$\partial_x \psi = -\begin{pmatrix} q^\dagger_x q - q_x q^\dagger & q^\dagger_{xx} + 2q^\dagger q q^\dagger \\ -q_{xx} - 2qq^\dagger q & -qq^\dagger_x + q_x q^\dagger \end{pmatrix}\psi \qquad \partial_t \psi = \begin{pmatrix} 0 & q^\dagger \\ -q & 0 \end{pmatrix}\psi. \tag{52}$$

Finally, we obtain the equation

$$q_{xxx} + 6q_x|q|^2 + q_t = 0, \qquad q^\dagger_{xxx} + 6q^\dagger_x|q|^2 + q^\dagger_t = 0. \tag{53}$$

This zero curvature equation can be expressed in two linear pairs of equations, namely

$$\psi_x = \begin{pmatrix} 0 & q^\dagger \\ -q & 0 \end{pmatrix}\psi \qquad \psi_t = \begin{pmatrix} q^\dagger_x q - q_x q^\dagger & q^\dagger_{xx} + 2q^\dagger q q^\dagger \\ -q^\dagger_{xx} - 2q^\dagger q q^\dagger & -q^\dagger_x q + q_x q^\dagger \end{pmatrix}\psi.$$

We can derive the KdV equation from this construction.

Corollary 5.2. *Let*

$$Q = \begin{pmatrix} 0 & u \\ -1 & 0 \end{pmatrix}; \tag{54}$$

then equation (50) yields the KdV equation

$$u_{xxx} + 6u_x|u + u_t = 0. \tag{55}$$

5.3. Nonholonomic deformation of KdV and coupled KdV equations via loop algebra

Let us now add terms

$$\nabla H_1 = (B_2\lambda^{-2} + B_1\lambda^{-1}) \tag{56}$$

to the gradient of Hamiltonian ∇H. Then the gradient of the modified Hamiltonian $\nabla\tilde{H}$ is given by

$$\nabla\tilde{H} = \underbrace{A\lambda^3 + Q\lambda^2 + \left(\frac{\mathrm{i}}{2}Q^+_x - \frac{\mathrm{i}}{2}Q^-_x - \frac{\mathrm{i}}{2}[Q^-, Q^+]\right)\lambda + \mathcal{W}}_{\text{original part}} + \underbrace{B_2\lambda^{-2} + B_1\lambda^{-1}}_{\text{deformation part}}, \tag{57}$$

where

$$\mathcal{W} = \left(-\tfrac{1}{4}Q_{xx} + \tfrac{1}{4}[Q^+, [Q^-, Q^+]] - \tfrac{1}{4}[Q^-, [Q^-, Q^+]]\right) + \left[\tfrac{1}{4}\left(Q^+_x + Q^-_x\right) + c(Q_+ + Q_-), Q\right]$$

with $\nabla\tilde{H} = \nabla H + \nabla H_1$, $\nabla H \in \Omega\mathfrak{g}^+$ and $\nabla H \in \Omega\mathfrak{g}^-$. Then the modified AKS equation for nonholonomic deformation is

$$\dot{\mu} = \left[\frac{\mathrm{d}}{\mathrm{d}x} + \mu, \nabla H\right] + \left[\frac{\mathrm{d}}{\mathrm{d}x} + \mu, \nabla H_1\right], \qquad \text{where} \qquad \mu = A\lambda + Q, \tag{58}$$

in which the second part of the rhs of equation (58) is the deformation part. Now compute the second expression $[\frac{\mathrm{d}}{\mathrm{d}x} + \mu, \nabla H_1]$; we obtain three additional equations:

$$\lambda^{-2} : B_{2x} + [Q, B_2] = 0, \tag{59}$$

$$\lambda^{-1} : B_{1x} + [A, B_2] + [Q, B_1] = 0, \tag{60}$$

$$\lambda^0 : Q_t = [A, B_1]. \tag{61}$$

Let us choose

$$B_1 = \frac{1}{2}\begin{pmatrix} -\mathrm{i}w & \mathrm{i}w_x \\ 0 & \mathrm{i}w \end{pmatrix}, \qquad B_2 = \frac{1}{2}\begin{pmatrix} w_x/2 & \int^x uw_x\,\mathrm{d}x' \\ w & -w_x/2 \end{pmatrix}. \tag{62}$$

The first two equations yield

$$\frac{w_{xx}}{2} + uw + \int^x uw_x\,\mathrm{d}x' = 0, \tag{63}$$

which after differentiation w.r.t. x yields

$$w_{xxx} + 4uw_x + 2u_x w = 0.$$

Hence, we recover the known constraint equation. Equating the λ^0 power term yields

$$u_t = -w_x. \tag{64}$$

This is the contribution coming from the deformation part of $\nabla\tilde{H}$. If take full $\nabla\tilde{H}$ into account, we obtain the KdV6 equation.

Proposition 5.3. *The KdV6 equation is a nonholonomic deformation of the Hamiltonian flow $\dot{\mu} = [\mu, \nabla\tilde{H}] + (\nabla\tilde{H})_x$ where $\nabla\tilde{H}$ is given by (57), $\mu = \lambda A + Q$ and matrices Q and B_i are given in (54) and (62) respectively.*

Thus, we give an alternative derivation of the KdV6 equation using the loop algebra technique.

5.4. Nonholonomic deformation of the coupled complex KdV equation

We come to our last section; here we apply the same technique to construct the nonholonomic deformation of the coupled complex KdV equation. Let us assume that Q and B_1 are given as before and B_2 is defined as

$$B_2 = \begin{pmatrix} w_x/2 & \int^x q^\dagger w_x\,\mathrm{d}x' \\ \int^x qw\,\mathrm{d}x' & -w_x/2 \end{pmatrix}.$$

Equating (59) and (60), we obtain two constraint equations of w

$$w_{xx} + 2\int^x q^\dagger w_x\,\mathrm{d}x' + 2q^\dagger w = 0, \tag{65}$$

$$w_{xx} + 2q\int^x q^\dagger w_x\,\mathrm{d}x' + 2q^\dagger\int^x qw_x = 0, \tag{66}$$

respectively. One must note that in this complex case, we have two different constraint equations which reduce to a single equation $w_{xxx} + 4uw_x + 2u_x w = 0$ for $q = 1$ and $q^\dagger = u$. Hence, equations (65) and (66) can be considered as a natural generalization of the stabilizer equation for a complex setting. Using the Adler–Kostant–Symes equation, we obtain the nonholonomic complex coupled KdV equation

$$q_t = q_{xxx} + 6q_x|q|^2 - w_x \tag{67}$$

$$q_t^\dagger = q_{xxx}^\dagger + 6q_x^\dagger|q|^2 - w_x, \tag{68}$$

where w satisfies (65) and (66). Let us state our result in the following form.

Proposition 5.4. *The nonholonomic deformation of the Hamiltonian flow corresponding to the Hamiltonian function $H(L) = -\frac{1}{2}\operatorname{tr}(L^2\lambda^{-2})$ and the initial condition $\nabla H(0)$ generates the system of the nonholonomic coupled KdV equation having the Lax representation*

$$\dot{\mu} = [\mu, L] + L_x,$$

where

$$
\begin{aligned}
L = \nabla H = A\lambda^3 + Q\lambda^2 &+ \left(\frac{\mathrm{i}}{2}Q_x^+ - \frac{\mathrm{i}}{2}Q_x^- - \frac{\mathrm{i}}{2}[Q^-, Q^+]\right)\lambda \\
&+ \left(-\frac{1}{4}Q_{xx} + \frac{1}{4}[Q^+, [Q^-, Q^+]] - \frac{1}{4}[Q^-, [Q^-, Q^+]]\right) \\
&+ \left[\frac{1}{4}(Q_x^+ + Q_x^-) + c(Q_+ + Q_-), Q\right] + B_2\lambda^{-2} + B_1\lambda^{-1},
\end{aligned}
$$

where B_2 *and* B_1 *are given by*

$$
B_1 = \frac{1}{2}\begin{pmatrix} -\mathrm{i}w & \mathrm{i}w_x \\ 0 & \mathrm{i}w \end{pmatrix}, \qquad B_2 = \frac{1}{2}\begin{pmatrix} w_x/2 & \mathrm{i}\int^x q^\dagger w_x\,\mathrm{d}x' \\ \int^x q w_x\,\mathrm{d}x' & -w_x/2 \end{pmatrix}.
$$

Thus, we give a derivation of another nonholonomic deformation of the coupled KdV-type equation.

6. Outlook

The present work focused on the construction of the nonholonomic deformation of several integrable systems. We have shown in this paper that the KdV6 equation has many equivalent representations. Starting from the coadjoint action of the extended Virasoro algebra, we derived explicit representation of the nonholonomic deformation of the Ito system, and using the loop algebra method we derived the nonholonomic deformation of the coupled complex KdV equations. In this way, we have discovered several new integrable partial differential equations belonging to the KdV family and this tells us that the domain of integrability still possesses hidden treasures. Following Kupershmidt, we have constructed an infinite number of conserved quantities of the Ito6 system. We have also shown that the Adler–Kostant–Symes scheme provides a geometrical method for constructing the nonholonomic deformation of the coupled complex KdV equation.

This work opens up various generalizations of nonholonomic deformed integrable systems. Extension of the construction and integrability properties of the nonholonomic deformation of the super KdV-type systems is under investigation. It would be a really challenging problem to construct the (2 + 1)-dimensional nonholonomic deformed systems. The Adler–Kostant–Symes scheme works elegantly in (1 + 1)-dimensional integrable systems but there is no unified approach to (2 + 1)-dimensional systems; moreover, most of these systems are not bi-Hamiltonian in nature. It would be interesting to study the properties of the Ito6 equation associated with the Schrödinger spectral problem as suggested in [5].

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